

AN INTERPOLATION PROBLEM FOR THE NORMAL BUNDLE OF CURVES OF GENUS $g \geq 2$ AND HIGH DEGREE IN \mathbb{P}^r

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ABSTRACT. Let $C \subset \mathbb{P}^n$ be a smooth curve and N_C its normal bundle. N_C satisfies strong interpolation if for all integers $s > 0$ and $\lambda_i \in \{0, 1, \dots, n-1\}$, $1 \leq i \leq s$, there are distinct points $P_1, \dots, P_s \in C$ and linear subspaces $U_i \subseteq E|_{P_i}$ such that $\dim(U_i) = \lambda_i$ for all i and the evaluation map $H^0(E) \rightarrow \bigoplus_{i=1}^s U_i$ has maximal rank (A. Atanasios). We prove that C satisfies strong interpolation if either C is a linearly normal elliptic curve or C is a general embedding of degree $d \geq (5n-8)g + 2n^2 - 5n + 4$ of a smooth curve X of genus $g \geq 2$.

1. INTRODUCTION

Let $C \subset \mathbb{P}^n$, be a smooth and connected projective curve. Set $d := \deg(C)$ and $g := p_a(C)$. Let N_C denote the normal bundle of C in \mathbb{P}^n . The sheaf N_C is a rank $r-1$ vector bundle on C and $\deg(N_C) = (r+1)d + 2g - 2$. Riemann-Roch gives $\chi(N_C) = (r+1)d - (r-3)(g-1)$. If (as always in this note) $h^1(\mathcal{O}_C(1)) = 0$, then $h^1(N_C) = 0$ and hence $h^0(N_C) = (r+1)d - (r-3)(g-1)$. In [1] A. Atanasov defined several interpolation problems for a vector bundle E on a smooth curve C . We assume $h^0(C, E) > 0$ and set $r := \text{rank}(E)$, $a := \lfloor h^0(E)/r \rfloor$ and $b := h^0(E) - ra$. E is said to satisfy *strong interpolation* if for all integers $s > 0$ and $\lambda_i \in \{0, 1, \dots, r\}$, $1 \leq i \leq s$, there are distinct points $P_1, \dots, P_s \in C$ and linear subspaces $U_i \subseteq E|_{P_i}$ such that $\dim(U_i) = \lambda_i$ for all i and the evaluation map $H^0(E) \rightarrow \bigoplus_{i=1}^s U_i$ has maximal rank, i.e., it is surjective if $\sum_i \lambda_i \leq h^0(E)$ and it is injective if $\sum_i \lambda_i \geq h^0(E)$. If these conditions are satisfied only for $s = a$ and $\lambda_i = r$ for all i (resp. only for $s = a+1$ and $\lambda_i = r$ for $i \leq a$, $\lambda_{a+1} = b$), then E is said to satisfy *weak interpolation* (resp. *regular interpolation*). Regular and strong interpolation are equivalent ([1], Theorem 8.1). In [1] this notion was applied to the case of the normal bundle N_C of a smooth curve $C \subset \mathbb{P}^n$. Many curves are proved to have normal bundle with strong interpolation or with weak interpolation ([1]). In this note we add other curves to the list.

Theorem 1. *Let X be a smooth curve of genus $g \geq 2$. Fix integers $n \geq 3$ and $d \geq (5n-8)g + 2n^2 - 5n + 4$. Let $C \subset \mathbb{P}^n$ be a general degree d embedding of X . Then N_C satisfies strong interpolation.*

Proposition 1. *Let $C \subset \mathbb{P}^n$, $n \geq 2$, be a linearly normal elliptic curve. Then N_C satisfies strong interpolation.*

2010 *Mathematics Subject Classification.* 14H50; 14H60.

Key words and phrases. normal bundle; curve in projective spaces.

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

In the case $r = 3$ the question of the existence of $C \subset \mathbb{P}^r$ with N_C satisfying interpolation seems to be quite different. A stronger condition is the condition $h^1(N_C(-2)) = 0$ (if $h^1(N_C) = 0$ it corresponds to require that the restriction map $H^0(E) \rightarrow E|S$ is bijective for a general $S \in |\mathcal{O}_C(2)|$). Let $\Gamma \subset \mathbb{N}^2$ be the set of all pairs $(d, g) \in \mathbb{N}^2$ such that there is a smooth, connected and non-degenerate curve $C \subset \mathbb{P}^3$ with $\deg(C) = d$, $p_a(C) = g$ and $h^i(N_C(-2)) = 0$, $i = 0, 1$. Several results on the set Γ are known ([2], [5], [6], [8], [9]). Here we just point out one of these results ([7], Corollaire 5.18). Fix any integer $g \geq 2$. Let $D(g)$ be the minimal integer x such that for all $d \geq x$ there is a smooth and connected projective curve $C \subset \mathbb{P}^3$ such that $\deg(C) = d$, $p_a(C) = g$ and $h^i(N_C(-2)) = 0$, $i = 0, 1$. It is well-known that if $d \gg 0$, then there is a smooth and connected curve $C \subset \mathbb{P}^3$ with $\deg(C) = d$, $p_a(C) = 0$ and $h^i(N_C(-2)) = 0$, $i = 0, 1$. Hence $D(g)$ is a well-defined integer. We have $\limsup D(g)/g^{2/3} = (9/8)^{1/3}$ ([7], Corollaire 5.18). D. Perrin also studied the h^0 -stability of the normal bundle of space curves ([7], §3).

2. THE PROOFS

Lemma 1. *Let E be a rank r vector bundle on C . Set $a := \lfloor h^0(E)/r \rfloor$ and $b := h^0(E) - ar$. Fix general subsets S, S' of C such that $\sharp(S) = a$ and $\sharp(S') = a + 1$. The vector bundle E satisfies strong interpolation if and only if $h^0(E(-S)) = b$ and $h^0(E(-S')) = 0$.*

Proof. The “only if” part is obvious. Assume $h^0(E(-S)) = b$ and $h^0(E(-S')) = 0$ for general S, S' . Fix $P \in S'$ and set $A := S' \setminus \{P\}$. Since S' is general, A is general and hence $h^0(E(-A)) = b$. The set A shows that E satisfies weak interpolation and that the restriction map $u : H^0(E) \rightarrow \oplus_{Q \in A} E|P$ is surjective. Hence the kernel V of u has dimension b . Since $h^0(E(-S')) = 0$, the restriction map $v : H^0(E) \rightarrow \oplus_{Q \in A} E|Q \oplus E|P$ is injective. Hence $v(V)$ is a b -dimensional linear subspace of $E|P$ and the restriction map $H^0(E) \rightarrow \oplus_{Q \in A} E|Q \oplus v(V)$ is bijective. Hence E satisfies regular interpolation. Hence E satisfies strong interpolation ([1], Theorem 8.1). \square

For all integers $r > 0$ and t and any smooth curve X of genus $g \geq 2$ let $M(X; r, t)$ denote the moduli space of all stable vector bundles on X with degree t and rank r . The scheme $M(X; r, t)$ is non-empty and irreducible.

Lemma 2. *Fix a general $E \in M(X; r, t)$. Then either $h^0(E) = 0$ or $h^1(E) = 0$. In both cases E satisfies strong interpolation.*

Proof. Write $t = r(g-1) + e$. By Riemann-Roch to check that $h^0(E) \cdot h^1(E) = 0$ we need to prove that $h^0(E) = \max\{0, e\}$. Riemann-Roch gives $h^0(E) \geq \max\{0, e\}$. Fix general line bundles L_1, \dots, L_r on X with $\deg(L_i) = g-1$ if $i < r$ and $\deg(L_r) = g-1+e$. We have $h^0(L_i) = 0$ if $i \neq r$ and $h^0(L_r) = \max\{0, e\}$. Since $g \geq 2$, every vector bundle on X is a flat limit of a family of stable vector bundles on X . The semicontinuity theorem for cohomology gives $h^0(E) \leq h^0(L_1 \oplus \dots \oplus L_r) = \max\{0, e\}$ and hence $h^0(E) = \max\{0, e\}$. To prove that E satisfies strong interpolation we may assume $e > 0$. Set $f := \lfloor e/r \rfloor$ and $h := e - rf$. Fix general subset S, S' of X with $\sharp(S) = f$ and $\sharp(S') = f+1$. Let R_i , $1 \leq i \leq h$, be a general line bundle on X of degree $g-1+f+1$ and let R_j , $h+1 \leq j \leq r$, be a general line bundle on X with degree $g-1+f$. Set $F := R_1 \oplus \dots \oplus R_r$. We have $h^0(F) = e$. Since F is a flat limit of a family of stable vector bundles on X , it is sufficient to

prove that $h^0(F(-S)) = h$ and $h^0(F(-S')) = 0$ (Lemma 1). This is true, because $h^0(R) = \max\{0, x - g + 1\}$ for a general $R \in \text{Pic}^x(X)$. \square

Proof of Theorem 1: For each integer $d \geq \max\{2g + 1, g + n\}$ let $A(n, d)$ be the set of all degree d embeddings of X into \mathbb{P}^n . This set is irreducible and non-empty. For each $f \in A(n, d)$ we get a vector bundle $f^*(N_{f(X)})$ on X . For a general f and d sufficiently large it is easy to check that $f^*(N_{f(X)})$ is stable. Since d is huge we even know that a general $E \in M(X; n - 1, (n + 1)d + 2g - 2)$ arises in this way ([3]). Lemma 2 gives that it satisfies strong interpolation. \square

Proof of Proposition 1: The case $n = 2$ is trivial, because N_C is a line bundle in this case. Assume $n \geq 3$. The case $i = 1$ of [4], Theorem 4.1, gives that N_C is poly-stable, i.e. it is a direct sum of stable vector bundles, all with the same slope $(n + 1)^2/(n - 1)$. We have $h^0(N_C) = (n + 1)^2$. Write $N_C \cong E_1 \oplus \cdots \oplus E_s$ with each E_i a stable vector bundle. Assume for the moment $n = 3$. In this case C is a complete intersection of two quadric surfaces and $N_C \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$. We have $h^0(C, N_C(-S)) = 0$ for any subset $S \subset C$ with $\sharp(S) = 8$ and S not the complete intersection of C with a quadric surface. Now assume that either $n = 4$ or $n \geq 6$. In this case each E_i has rank at least two, because $(n + 1)^2/(n - 1) = n + 3 + 4/(n - 1) \notin \mathbb{Z}$. Set $a := \lfloor (n + 1)^2/(n - 1) \rfloor$. Take any subset S, S' of C such that $\sharp(S) = a$ and $\sharp(S') = a + 1$. Since E_i is stable, $E_i(-S)$ and $E_i(-S')$ are stable. Since $\deg(E_i(-S)) > 0$ and $E_i(-S)$ is stable and not a line bundle, duality gives $h^1(E_i(-S)) = 0$. Hence $h^1(N_C(-S)) = 0$, i.e. $h^0(N_C(-S)) = (n + 1)^2 - (n - 1)a$. Hence N_C satisfies weak interpolation. Since $\deg(E_i(-S')) < 0$ and $E_i(-S')$ is stable, we have $h^0(E_i(-S')) = 0$. Lemma 1 gives that N_C satisfies strong interpolation. Now assume $n = 5$. In this case each E_i is a line bundle and strong and weak interpolation are equivalent. Let $B \subset C$ be any subset of C with $\sharp(B) = 9$. We have $h^0(N_C(-S)) = 0$ if and only if B is not a divisor of one of the linear systems $|E_i|$, $1 \leq i \leq 5$. \square

Remark 1. The proof of Proposition 1 shows that if either $n = 4$ or $n \geq 6$ for any zero-dimensional scheme $Z \subset C$ the restriction map $H^0(E) \rightarrow E|_Z$ has maximal rank.

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